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Doctor of Science, Professor **Alexander Nitsyn**,
alnitsyn@gmail.com, ORCID 0000-0001-7900-2612
National Technical University «Kharkov Polytechnic Institute»

PARQUETS FROM REGULAR FIVE-POINTED AND TEN-POINTED STARS

Obviously, regular pentagons cannot fill a plane without overlaps and gaps. However, the plane can be filled without overlaps and gaps with polygons whose angles are multiples of 36°. For example, if a regular pentagon is changed into a regular five-pointed star, composed of five rhombuses with angles of 72° and 108°, and supplemented with five rhombuses with angles of 36° and 144° to form a regular decagon, then the resulting rhombuses can fill the plane without overlaps and gaps.

It is assumed that, in addition to the presented method of tiling the plane with the rhombuses pointed out above, there are other ways of tiling the plane without overlaps and gaps with polygons whose angles are multiples of 36°. This explains why the challenge of tiling a plane with regular pentagons attracts the attention of not only geometers, but also designers who create new types of ornaments. Moreover, the regular pentagon among other types of regular polygons has the highest aesthetic qualities, and parquets made up of polygons, the angles of which are multiples of 36°, surpass other types of parquet in beauty and perfection. Therefore, the working out of methods for tiling a plane without overlaps and gaps with polygons, the angles of which are multiples of 36°, is an actual challenge for both geometers and designers creating new types of ornaments.

For the first time, two variants of the parquet, composed of rhombuses forming five-pointed and ten-pointed stars, were worked out. If in the first variant the center of the parquet is a five-pointed star, then in the second variant it is a ten-pointed star. Moreover, if in the first variant the parquet does not have a single plane of symmetry, then in the second variant the parquet has twenty planes of symmetry. Another difference is that if in the first variant the parquet has a rotational symmetry with a 5th order symmetry axis, then in the second variant it has a rotational symmetry with a 10th order symmetry axis. Common to both variants of parquet is that they belong to non-periodic parquets, that is, they are new, previously unexplored types of Penrose mosaics. It is assumed that our further research will be directed to the invention of parquet, which has neither translation nor rotation symmetry, and at the same time maintains order in the arrangement of tiles.

Key Words: mosaics; parquets; translational symmetry; non-periodic tiling of the plane; regular five-pointed and ten-pointed stars.

Formulation of the problem. Let's take a dodecahedron – a regular polyhedron, the surface of which consists of twelve regular pentagons. Cut its lateral surface in such a way that its edges are the cut lines, and after superposing its faces with the plane, we will make sure that it is impossible to fill the plane with regular pentagons without overlaps and gaps.

However, the plane can be filled without overlaps and gaps with polygons whose angles are multiples of *36°*. For example, if a regular pentagon is changed into a regular five-pointed star, composed of five rhombuses with angles of *72°* and *108°*, and supplemented with five rhombuses with angles of *36°* and *144°* to form a regular decagon, then the resulting rhombuses can fill the plane without overlaps and gaps.

We assume that, in addition to the presented method of tiling the plane with the rhombuses pointed out above, there are other ways of tiling the plane without overlaps and gaps with polygons whose angles are multiples of 36°. This explains why the challenge of tiling a plane with regular pentagons attracts the attention of not only geometers, but also designers who create new types of ornaments. Moreover, the regular pentagon among other types of regular polygons has the highest aesthetic qualities, and parquets made up of polygons, the angles of which are multiples of 36°, surpass other types of parquet in beauty and perfection. Therefore, the working out of methods for tiling a plane without overlaps and gaps with polygons, the angles of which are multiples of *36°*, is an actual challenge for both geometers and designers creating new types of ornaments.

Analysis of recent research and publications. It should be said that in the foreign literature there are many works devoted to parquets composed of rhombuses with angles of *72°* and *108°* and rhombuses with angles of *36°* and *144°*, for example, Penrose mosaics [1–6]. Unfortunately, there are no such works in the domestic literature at all. Everything we know about Penrose mosaics, we know from translated literature [7, 8]. However, Roger Penrose studied far from all types of parquet that can be made up of rhombuses with angles of *72°* and *108°* and rhombuses with angles of *36°* and *144°*. For example, Penrose mosaics are variants of a parquet composed in such a way that rhombuses with angles of *72°* and *108°* is assembled into regular five-pointed stars, and rhombuses with angles of *36°* and *144°* fill the gaps between them. Parquets, composed in such a way that rhombuses with angles of *72°* and *108°* form regular five-pointed stars, and rhombuses with angles of *36°* and 144° – regular ten-pointed stars, have not yet been invented by any researcher, including Roger Penrose. Therefore, before geometers, desiring to discover new types of parquet, composed of rhombuses with angles of *72°* and *108°* and rhombuses with angles of *36°* and *144°*, a wide field of activity opens up.

Thus, the **purpose of the study** is to work out a method for tiling a plane in such a way that the rhombuses with angles of *72°* and *108°* form regular five-pointed stars, and rhombuses with angles of *36°* and *144°* – regular ten-pointed stars.

Main part. Let's build a regular pentagon. Change it into a regular five-pointed star, made up of five rhombuses, the angles of which are *72°* and *108°*. Take a rhombus with angles of *72°* and *108°*. Let's rotate it around an axis passing through the vertex of the rhombus with an angle of *72°* and perpendicular to its plane, by an angle of *72°*. Repeat the rotation transformation three more times and get a regular five-pointed star, composed of five rhombuses with angles of *72°* and *108°*. Note that a regular five-pointed star has five planes of symmetry, that is, it has a reflection symmetry group of the 5th order. Moreover, a regular five-pointed star has rotational symmetry with a 5th order symmetry axis. This means that it can be superposed with itself by rotating around the axis of symmetry by an angle of *72°*.

Supplement the regular five-pointed star with five rhombuses with angles of *36°* and *144°*. We get a regular decagon filled without gaps and overlaps with five rhombuses with angles of *72°* and *108°* and five rhombuses with angles of *36°* and *144°*. Let's present the resulting geometric figure as a regular five-pointed star inscribed in a regular decagon.

The regular decagon, which includes a regular five-pointed star, is remarkable in that its repetitions can completely fill the plane. Moreover, in some cases, regular decagons are connected in such a way that a rhombus with angles of *36°* and *144°* of one geometric figure is superimposed on the same rhombus of another geometric figure, and the gaps between them are filled with rhombuses with angles of *72°* and *108°*.

Take a rhombus with angles of *36°* and *144°*. Let's rotate it around an axis passing through the vertex of the rhombus with an angle of *36°* and perpendicular to its plane, by an angle of *36°*. Repeat the rotation transformation eight more times and get a regular ten-pointed star, composed of five rhombuses with angles of *72°* and *108°*. Note that a regular ten-pointed star has five planes of symmetry, that is, it has a reflection symmetry group of the 10th order. , a regular ten-pointed star has rotational symmetry with a 10th order symmetry axis. This means that it can be superposed with itself by rotating around the axis of symmetry by an angle of 36°.

The regular ten-pointed star, composed of ten rhombuses with angles of *36°* and *144°*, is remarkable in that its repetitions can completely fill the plane. Additionally, the regular ten-pointed stars are placed in such a way that the gaps between them are filled in a certain order with five-pointed stars and rhombuses with angles of *72°* and *108°*.

Let's show in Fig. 1 the first variant of the parquet, which is formed by tiling the plane with a geometric figure, which includes a regular ten-pointed star, made up of ten rhombuses with angles of 36° and 144°.

Consider the types of symmetry possessed by a geometric figure, which includes a regular ten-pointed star, composed of ten rhombuses with angles of 36° and 144°, and parquet constructed on its basis according to the first variant.

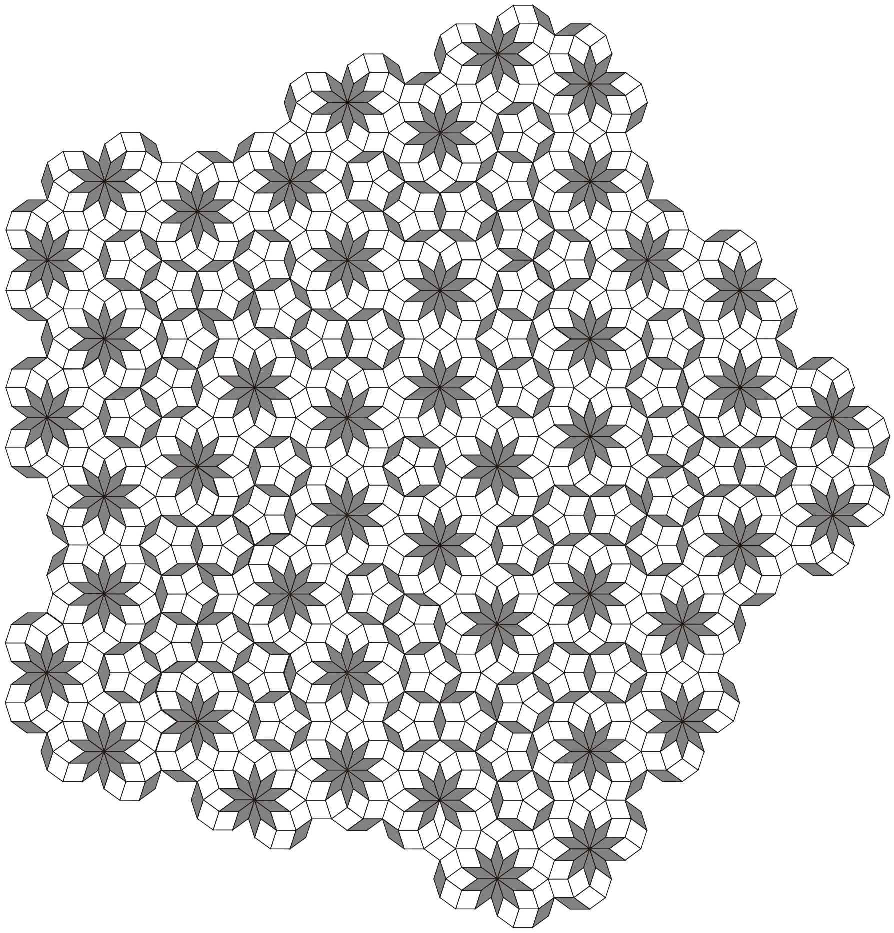

Fig. 1. The first variant of the parquet, which is formed by tiling the plane with regular five-pointed stars and regular ten-pointed stars, made up of ten rhombuses with angles of 36° and 144°

Obviously, a geometric figure, which includes a regular ten-pointed star, composed of ten rhombuses with angles of 36° and 144°, has ten planes of symmetry, that is, it has a reflection symmetry group of the 10th order. The parquet shown in Fig. 1 does not have any plane of symmetry, that is, it does not have any reflection symmetry group. Moreover, it has a rotational symmetry with a 5th order symmetry axis. This means that the parquet we are considering can be superposed with itself by rotation around the axis of symmetry by an angle of 72°. However, the parquet shown in Fig. 1 does not possess the translational symmetry. Consequently, it cannot be superposed with itself by means of a parallel translation in any direction specified by the translation axis. Therefore, it can be affirmed that the parquet we are considering is non-periodic parquet, that is, another type of Penrose mosaic that has not been studied before.

It is noteworthy that the parquet shown in Fig. 1 does not exhaust all variants for tiling the plane with a geometric figure, which includes a regular ten-pointed star made up of ten rhombuses with angles of *36°* and *144°*.

Let's show in Fig. 2 the second variant of the parquet, which is formed by tiling the plane with a geometric figure, which includes a regular ten-pointed star, made up of ten rhombuses with angles of *36°* and *144°*.

Consider the types of symmetry possessed by a geometric figure, which includes a regular ten-pointed star, composed of ten rhombuses with angles of *36°* and *144°*, and parquet constructed on its basis according to the second variant.

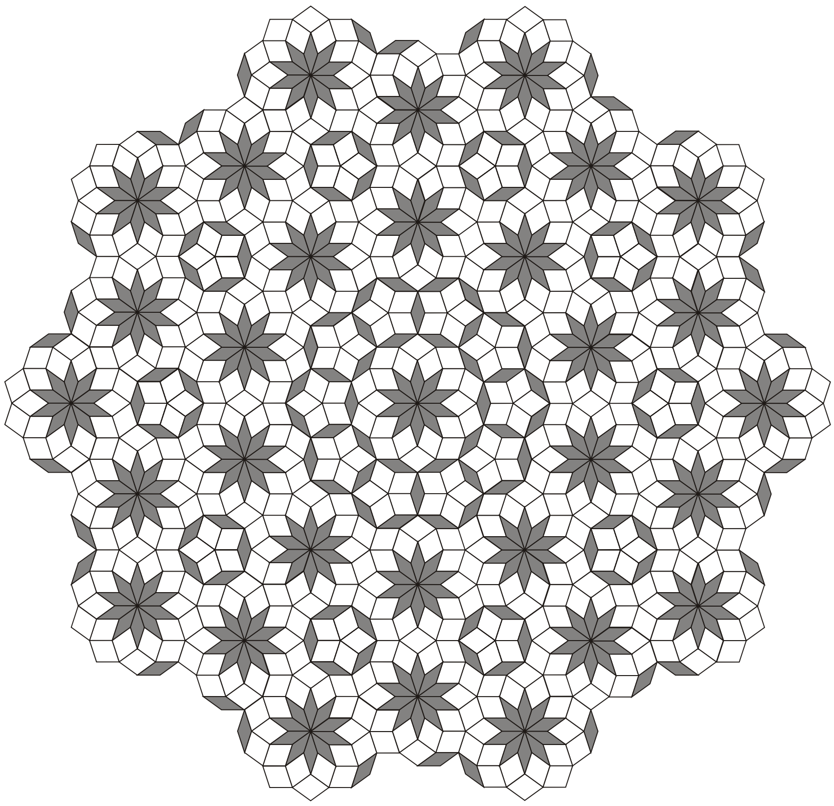

Fig. 2. The second variant of the parquet, which is formed by tiling the plane with regular five-pointed stars and regular ten-pointed stars, made up of ten rhombuses with angles of *36°* and *144°*.

Obviously, a geometric figure, which includes a regular ten-pointed star, composed of ten rhombuses with angles of *36°* and *144°*, has ten planes of symmetry, that is, it has a reflection symmetry group of the 10th order. The parquet shown in Fig. 2 has twenty planes of symmetry, that is, it has a reflection symmetry group of the 20th order. In addition, it has a rotational symmetry with a 10th order axis of symmetry. This means that the parquet we are considering can be combined with itself by rotation around the symmetry axis by an angle of *36°*. However, the parquet shown in Fig. 2 does not possess the translational symmetry. Consequently, it cannot be superposed with itself by means of parallel translation in any direction specified by the translation axis.

Therefore, it can be affirmed that the parquet we are considering is non-periodic parquet, that is, another type of Penrose mosaic that has not been studied before.

At the same time, it should be noted that, despite the fact that the regular ten-pointed star is a combination of ten rhombuses with angles of *72°* and *108°*, it is perceived as a solid geometric figure, similar to such geometric figures as a rhombus with angles of *72°* and *108°*, rhombus with angles of *36°* and *144°* and regular five-pointed star. Therefore, if we assume that Penrose mosaics is assembled of three geometric figures: a rhombus with angles of *72°* and *108°*, a rhombus with angles of *36°* and *144°*, and a regular five-pointed star, then it can be affirmed that the parquets shown in Fig. 1 and Fig. 2, is assembled of four geometric figures: a rhombus with angles of *72°* and *108°*, a rhombus with angles of *36°* and *144°*, a regular five-pointed star and a regular ten-pointed star. Since the geometric figure in the form of a ten-pointed star, composed of ten rhombuses with angles of *72°* and *108°*, has high aesthetic qualities, its introduction into the Penrose mosaic enriches it with new representational means and increases its artistic value.

Conclusions and prospects. Thus, for the first time, two variants of the parquet, composed of rhombuses forming five-pointed and ten-pointed stars, were worked out. If in the first variant the center of the parquet is a five-pointed star, then in the second variant it is a ten-pointed star. Moreover, if in the first variant the parquet does not have a single plane of symmetry, then in the second variant the parquet has twenty planes of symmetry. Another difference is that if in the first variant the parquet has a rotational symmetry with a 5th order symmetry axis, then in the second variant it has a rotational symmetry with a 10th order symmetry axis. Common to both variants of parquet is that they belong to non-periodic parquets, that is, they are new, previously unexplored types of Penrose mosaics. We assume that our further research will be directed to the invention of parquet, which has neither translation nor rotation symmetry, and at the same time maintains order in the arrangement of tiles.

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д. т. н., професор **Ніцин О. Ю.**,
alnitsyn@gmail.com, ORCID 0000-0001-7900-2612
Національний технічний університет
«Харківський політехнічний інститут»

ПАРКЕТИ З ПРАВИЛЬНИХ П'ЯТИКУТНИХ ТА ДЕСЯТИКУТНИХ ЗІРОК

Очевидно, що правильними п'ятикутниками не можна заповнити площину без накладень і пропусків. Однак площину можна заповнити без накладень та пропусків багатокутниками, кути яких є кратними 36°.

Наприклад, якщо правильний п'ятикутник перетворити на правильну п'ятикутну зірку, складену з п'яти ромбів з кутами 72° та 108° і доповнити її п'ятьма ромбами з кутами 36° і 144°, щоб утворився правильний десятикутник, отриманими ромбами можна заповнити площину без накладень та пропусків.

Припущено, що, крім представленого способу замощення площини зазначеними вище ромбами, існують інші способи замощення площини без накладень і пропусків багатокутниками, кути яких є кратними 36°. Це пояснює, чому задача про замощення площини правильними п'ятикутниками привертає увагу не лише геометрів, а й дизайнерів, що створюють нові види орнаментів. Крім того, правильний п'ятикутник серед інших видів правильних багатокутників має найвищі естетичні якості, а паркети, складені з багатокутників, кути яких є кратними 36°, перевершують інші види паркету красою та досконалістю. Тому розроблення способів замощення площини без накладень та пропусків багатокутниками, кути яких є кратними 36°, є нагальним завданням як геометрів, так і дизайнерів, що створюють нові види орнаментів.

Вперше розроблено два варіанти паркету, складеного з ромбів, що утворюють п'ятикутні та десятикутні зірки. Якщо у першому варіанті центром паркету є п'ятикутна зірка, то у другому варіанті десятикутна зірка. Крім того, якщо у першому варіанті паркет не має жодної площини симетрії, то у другому варіанті паркет має двадцять площин симетрії. Ще однією відмінністю є те, що якщо у першому варіанті паркет має симетрію обертання з віссю симетрії 5-го порядку, то у другому варіанті – симетрію обертання з віссю симетрії 10-го порядку. Спільним для обох варіантів паркету є те, що вони належать неперіодичним паркетам, тобто є новими, не вивченими раніше видами мозаїки Пенроуза. Припущено, що подальші дослідження будуть спрямовані на винахід паркету, що не має ні симетрії переносу, ні симетрії обертання і водночас зберігає закономірність розташування плиток.

Ключові слова: мозаїки; паркети; симетрія переносу; неперіодичне замощення площини; правильні п'ятикутні та десятикутні зірки.