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PARQUETS BASED ON A FRACTAL EXTENSION OF A REGULAR PENTAGON

Consider the development of a dodecahedron - a regular polyhedron, the surface of which consists of twelve regular pentagons. Let’s represent the deployment of the dodecahedron in the form of two groups of polygons, consisting of 6 regular pentagons, one of which is located in the center of the group, while the others are adjacent to its sides. However, the most remarkable thing is that each group of 6 regular pentagons forms a figure that inscribes into a regular pentagon. It follows that each group of 6 regular pentagons can be considered as the result of a fractal expansion of the regular pentagon located in its center. Moreover, if we attach the same figure to each side of the figure into which a group of 6 regular pentagons inscribes, we will get another figure similar to the original regular pentagon. Repeat the previous steps infinitely many times and get a figure similar to a regular pentagon and completely filling the plane.

For the first time, a fractal extension of a regular pentagon was applied to the tiling of a plane, the gaps of which are eliminated by figures that are combinations of a rhombus with angles of 36° and 144°, a rhombus with angles of 72° and 108°, and a regular five-pointed star with an angle of 36°. It is shown that the variety of polygons used to eliminate gaps provides parquets in the form of a fractal extension of a regular pentagon with a higher artistic value compared to parquets constructed by means of a fractal extension of a square. It is presented a parquet variant, in which gaps between regular pentagons, formed after the first iteration, are filled with rhombuses with angles of 36° and 144°. It is shown that the figures that fill the gaps that appear after each iteration are similar to a rhombus with angles of 36° and 144°, and the parquet obtained after the fourth iteration and truncated by a regular pentagon is similar to a figure consisting of four rhombuses with angles of 72° and 108° and one regular five-pointed star with an angle of 36° and inscribed in a regular pentagon. Additionally, parquet variants are presented, in which the gaps between regular pentagons, formed after the first iteration, are filled with the corners of regular five-pointed stars with an angle of 36° at the top.

Key Words: mosaics, parquets of rhombuses and regular five-pointed stars, fractal extension of a regular pentagon.

Formulation of the problem. Why do some things seem beautiful to us and others not so much? We will not talk about all the fine arts, but we can definitely say about ornaments that they are beautiful, because they are law-governed. Indeed, the ornaments of all times and peoples are excellent visual aids for the study of all the laws of symmetry known to us. What kind of symmetry we do not meet in ornaments created by artists and artisans of Ancient Egypt and Ancient Persia, Ancient Greece and Byzantium, Ancient India and Ancient China! We read in the intricacies of lines and geometric figures the symmetry of reflection, the symmetry of rotation and the symmetry of transfer. In addition, we see in ancient ornaments not only special cases of symmetry, such as central symmetry, but also combinations of symmetry generated by reflections, rotations and translations, such as sliding symmetry. Of course, the unknown masters of antiquity did not hear and did not know in spirit about such tricky concepts as symmetry and plane transformations, but this did not prevent them from creating works of art that outlived them for centuries and millennia. Meanwhile, combinations of symmetry generated by reflections and rotations, as well as transfers, are still the subject of research in various fields of art and natural science today. For example, symmetry is widely used as one of the techniques for constructing borders - flat figures that have one or more translation symmetries in combination with reflection symmetries. Therefore, despite the fact that a person's passion for decorating household items with patterns arose in ancient times, the study of the laws of symmetry and the creation of new types of ornament on their basis still remains an urgent challenge for both geometers and designers who have devoted themselves to ornamental art.

Analysis of recent research and publications. At the same time, it should be noted that the foundations of the theory of symmetry were laid only a few decades ago thanks to the works of E. S. Fedorov (1853–1919), Hermann Weyl (1885–1955), A. V. Shubnikov (1887–1970), N. V. Belov (1891–1982), Harold Coxeter (1907–2003) and other prominent scientists of the 20th century [1–5]. In our opinion, a special place in the above list is occupied by the Russian crystallographer E. S. Fedorov, who discovered 17 types of symmetry, which are found not only in crystal lattices, but also in many works of art by the masters of the Ancient World and the Middle Ages. In connection with the concept of symmetry, it is impossible not to mention the remarkable artist and graphic artist Maurits Escher (1898–1972), who, thanks to the study of the ornaments of Ancient Persia and fruitful collaboration with such outstanding mathematicians as Harold Coxeter and Roger Penrose, embodied many laws of symmetry in drawings and prints. In addition, it would be a great sin not to recall the works of another outstanding geometer D. I. Tkach, who became famous for creating a wonderful fractal called 'Snowflakes of the Tkach-Nifanin', as well as a method for tiling a plane with tiles in the form of a gamma cross, colloquially referred to as “swastika” » [6–9]. Meanwhile, it must be said that the ‘Snowflakes of the Tkach-Nifanin’ is a curve that does not fill the entire plane. The application of the fractal extension of the square to the filing of the plane was not considered by D. I. Tkach.

Thus, the **purpose of the study** is to apply the fractal extension of the regular pentagon to the filing of the plane with regular polygons whose angles are multiples of *36°*, without overlaps and gaps.

Main part. Along with such wonderful fractals as the Koch Snowflake, Sierpinski Carpet and the Hilbert Curve, the Dürer Snowflake fractal, named after the great artist and graphic artist of the Northern Renaissance Albrecht Dürer (1471–1528), is widely known among geometers. This name is due to the fact that the figure, consisting of six regular pentagons and having the shape of a regular pentagon with five wedges removed from it, was invented by Albrecht Dürer and first described by him in the treatise ‘Guide to measuring with a compass and ruler ...’, published in 1525.

Here is an excerpt from the second book of the treatise: ‘Fifth, you can combine pentagons in the following manner. First draw a pentagon and place pentagons of the same size on each side. Then place five pentagons on their sides, particularly along the two sides. This will result in the formation of five narrow lozenges between them. Then add pentagons in the angles which will have formed, so that these will touch the narrow lozenges with their corners. You can continue in this manner as long as you desire’ [10, p. 99]. It is remarkable that today we call the figure invented by Albrecht Dürer as a fractal, and the method of its construction is a fractal extension of a regular pentagon.

We will show the application of the figure, compiled by Albrecht Dürer, to the fractal division of a regular pentagon [11–18]. Let’s take a regular pentagon and cut out 5 isosceles triangles from it, in which the height passes through the middle of each side of the regular pentagon, and the ratio of the larger side to the smaller side of the triangle is equal to the ‘golden section’. We will get a group of 6 regular pentagons, in the center of which there is a regular pentagon, and 5 regular pentagons adjoin each of its sides and form a figure reminding a five-pointed star. From each regular pentagon included in the group of 6 regular pentagons, we cut out 5 isosceles triangles similar to the isosceles triangle introduced when constructing the first iteration of the fractal division of the regular pentagon. Let’s repeat the above actions infinitely many times and get a figure similar to ‘Dürer’s Snowflake’.

Let’s show in Fig. 1 the opening four iterations of the fractal division of a regular pentagon in a manner based on the figure invented by Albrecht Dürer.

Thus, the discovery of the fractal extension of a regular pentagon belongs to Albrecht Dürer. However, in his treatise ‘Guide to measuring with a compass and a ruler ...’ does not say a word about the fact that the fractal expansion of a regular pentagon can completely fill the plane, if you invent a way by which you can eliminate the gaps that form after each iteration of the fractal expansion of a regular pentagon.

Let us apply the fractal extension of a regular pentagon to tiling the plane with regular polygons whose angles are multiples of 36°, without overlaps or gaps.

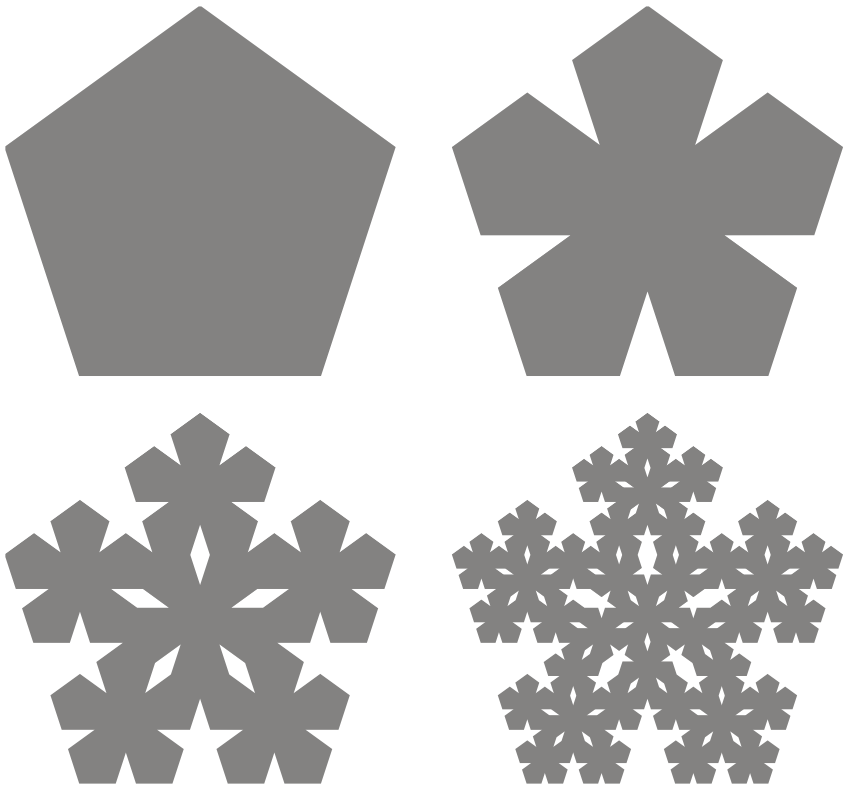

Fig. 1. The opening four iterations of the fractal division of a regular pentagon

Consider the development of a dodecahedron - a regular polyhedron, the surface of which consists of twelve regular pentagons. Let's represent the deployment of the dodecahedron in the form of two groups of polygons, consisting of 6 regular pentagons, one of which is located in the center of the group, while the others are adjacent to its sides. However, the most remarkable thing is that each group of 6 regular pentagons forms a figure that inscribes into a regular pentagon. It follows that each group of 6 regular pentagons can be considered as the result of a fractal expansion of the regular pentagon located in its center. Moreover, if we attach the same figure to each side of the figure into which a group of 6 regular pentagons inscribes, we will get another figure similar to the original regular pentagon. Repeat the previous steps infinitely many times and get a figure similar to the original regular pentagon and filling the plane with gaps that can be filled with regular polygons whose angles are multiples of 36°.

Thus, a regular pentagon has the remarkable property of forming a figure similar to itself and taking up an area 6 times larger than its area. We will call this property the fractal extension of a regular pentagon.

Let's show in Fig. 2 group of regular polygons that fill the gaps that are formed after four iterations of the fractal expansion of a regular pentagon.

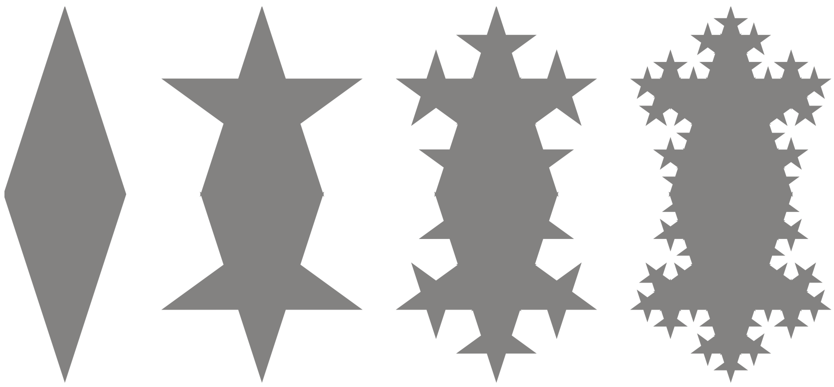

Fig. 2. Group of regular polygons that fill the gaps that are formed after four iterations of the fractal expansion of a regular pentagon

Let us fill the gaps formed between the groups of 6 and 36 regular pentagons with the figures shown in Fig. 2, and we get a figure reminding a five-pointed snowflake. Let us pick out from the resulting figure a fragment that inscribes into a regular pentagon and show it in Fig. 3.

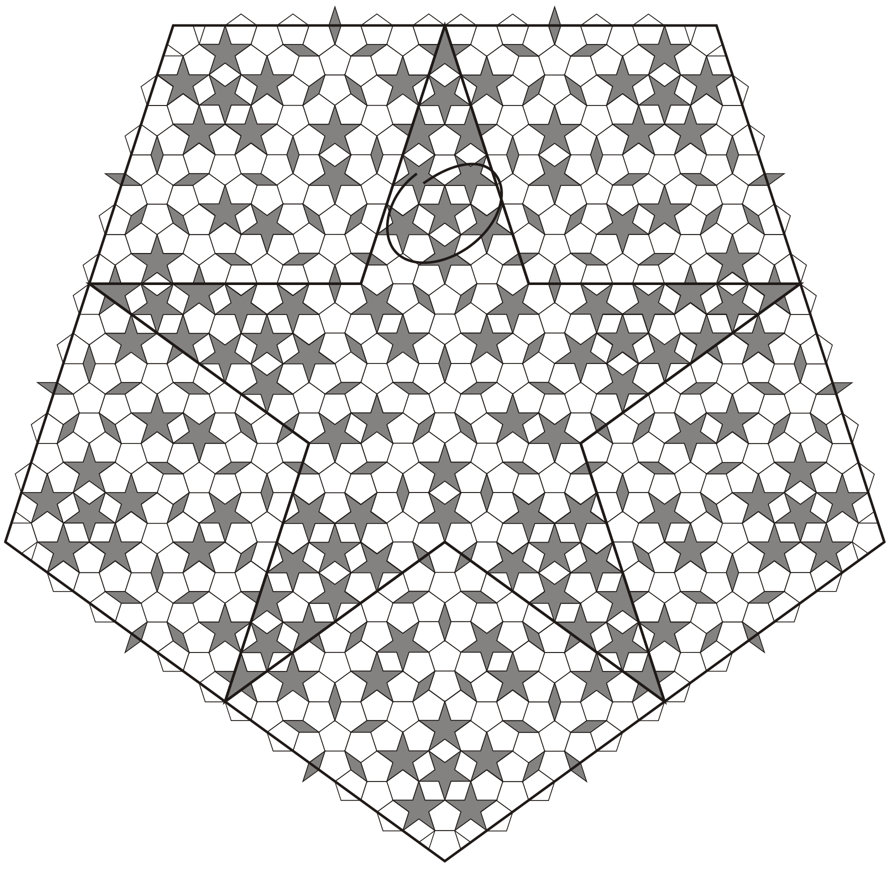

Fig. 3. The first variant of the parquet, forming with using the fractal expansion of a regular pentagon

Note that if we inscribe into the figure shown in Fig. 3, a regular five-pointed star with an angle of 36°, then we pick up four rhombuses with angles of 72° and 108° adjacent to it. It follows that the parquet obtained after the fourth iteration and truncated by a regular pentagon is similar to a figure consisting of four rhombuses with angles of 72° and 108° and one regular five-pointed star with an angle of *36°* and inscribed in a regular pentagon. This property of the parquet shown in Fig. 3 is remarkable in that the group of polygons that fill the gaps after the third iteration has been completed includes a figure also consisting of four rhombuses with angles of *72°* and *108°* and one regular five-pointed star with an angle of *36°*.

It follows that the choice of a regular five-pointed star with an angle of *36°* to fill the gaps formed after each an iteration of the fractal expansion of a regular pentagon, is not accidental and is predetermined by the geometric properties of the figure shown in Fig. 3.

Let's consider the types of symmetry that parquet constructed by means of a fractal extension of a regular pentagon according to the first variant, has.

Obviously, the parquet shown in Fig. 3 has ten planes of symmetry; consequently, it has a reflection symmetry group of the 10th order. In addition, it has a rotational symmetry with an axis of symmetry of the 5th order. This means that the parquet we are considering can be superposed with itself by rotating around the axis of symmetry by an angle of *72°*. However, the transfer symmetry of the parquet shown in Fig. 3 does not exist. It follows that it cannot be superposed with itself by means of parallel translation in any direction given by the translation axis.

Meanwhile, it is possible to obtain with the help of a fractal extension of a regular pentagon, not only parquet with a reflection symmetry group of the 10th order, but also parquet that does not have any plane of symmetry.

Let's present the second variant of the parquet constructed with the help of a fractal extension of a regular pentagon. This variant differs from the previous one in that the gaps between regular pentagons formed after the first iteration are filled not with rhombuses with angles of *36°* and *144°*, but with the corners of regular five-pointed stars with an angle of *36°*.

Let's show in Fig. 4 is the second variant of the parquet constructed by means of a fractal extension of a regular pentagon and obtained after three iterations.

Let's consider the types of symmetry that parquet constructed by means of a fractal extension of a regular pentagon according to the second variant, has.

Obviously, the parquet shown in Fig. 4 does not have any plane of symmetry; consequently, it does not have a reflection symmetry group. However, it has a rotational symmetry with a 5th order symmetry axis. This means that the parquet we are considering can be superposed with itself by rotating around the axis of symmetry by an angle of *72°*. At the same time, the transfer symmetry of the parquet shown in Fig. 4 does not exist. It follows that it cannot be superposed with itself by means of parallel translation in any direction given by the translation axis.

Let's present the third variant of the parquet constructed by means of a fractal extension of a regular pentagon. This variant differs from the previous one in that the step formed by the side of a regular pentagon, into which the figure obtained after the second iteration is inscribed, is eliminated by adding it by a group of polygons consisting of a rhombus with angles *36°* and *144°*, a rhombus with angles *72°* and *108°* and a regular five-pointed star with an angle of *36°*.

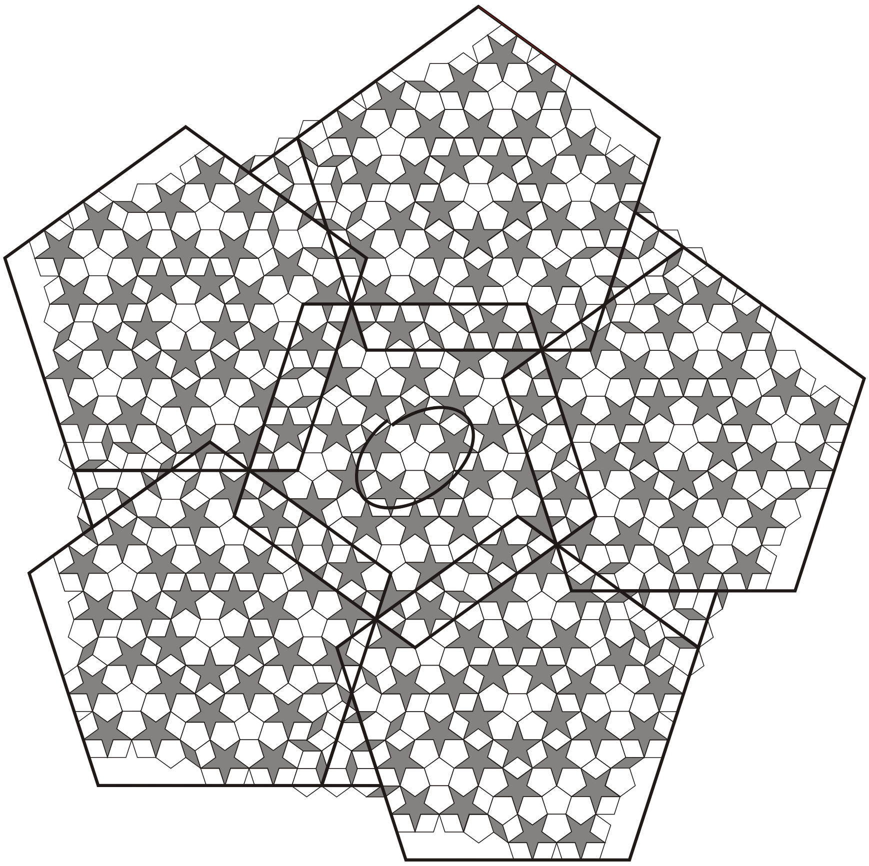

Fig. 4. Second variant of the parquet constructed by means of a fractal extension of a regular pentagon and obtained after three iterations

Let's show in Fig. 5 is the third variant of the parquet constructed by means of a fractal extension of a regular pentagon and obtained after three iterations.

Let's consider the types of symmetry that parquet constructed by means of a fractal extension of a regular pentagon according to the third variant, has.

Obviously, the parquet shown in Fig. 5 does not have any plane of symmetry; consequently, it does not have a reflection symmetry group. However, it has a rotational symmetry with a 5th order symmetry axis. This means that the parquet we are considering can be superposed with itself by rotating around the axis of symmetry by an angle of *72°*. At the same time, the transfer symmetry of the parquet shown in Fig. 5 does not exist. It follows that it cannot be superposed with itself by means of parallel translation in any direction given by the translation axis.

Let us show that the figure shown in Fig. 5 is a fractal. Let's inscribe in the groups of polygons that fill the gaps between the groups of 30 regular five-pointed stars formed after the third iteration, isosceles triangles, in which the ratio of the larger side to the smaller one is equal to the 'golden section'. We get a figure that has the shape of a regular pentagon with five wedges cut out of it. A remarkable property of the resulting figure is that its shape is similar to the group of 6 regular pentagons formed after the first iteration. It follows that the figure shown in Fig. 5 is a fractal.

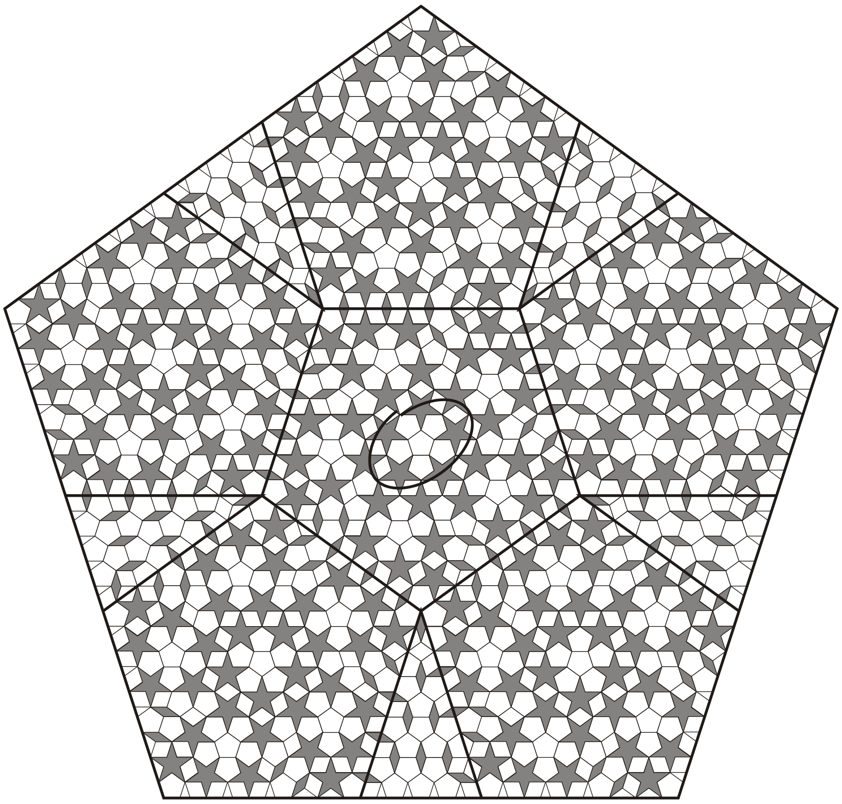

Fig. 5. Third variant of the parquet constructed by means of a fractal extension of a regular pentagon and obtained after three iterations

Consider the aesthetic qualities of the parquet presented in Fig. 3–Fig. 5. Note that if the plane is filled with polygons using the fractal expansion of a square, the gaps formed after each iteration can only be filled with squares that have different sides. In our opinion, this somewhat reduces the artistic value of the parquet constructed by means of a fractal expansion of the square. Meanwhile, when tiling the plane using a fractal extension of a regular pentagon, the gaps that form after each iteration are filled with figures that are a combination of a rhombus with angles of 36° and 144°, a rhombus with angles of 72° and 108°, and a regular five-pointed star with an angle of 36°. This, in our opinion, gives parquets constructed by means of a fractal expansion of a regular pentagon a higher artistic value compared to parquets constructed by means of a fractal expansion of a square.

Conclusions and prospects. Thus, for the first time, a fractal extension of a regular pentagon was applied to the tiling of a plane, the gaps of which are eliminated by figures that are combinations of a rhombus with angles of 36° and 144°, a rhombus with angles of 72° and 108°, and a regular five-pointed star with an angle of 36°. It is shown that the variety of polygons used to eliminate gaps provides parquets in the form of a fractal extension of a regular pentagon with a higher artistic value compared to parquets constructed by means of a fractal extension of a square. It is presented a parquet variant, in which gaps between regular pentagons, formed after the first iteration, are filled with rhombuses with angles of 36° and 144°. It is shown that the figures that fill the gaps that appear after each iteration are similar to a rhombus with angles of 36° and 144°, and the parquet obtained after the fourth iteration and truncated by a regular pentagon is similar to a figure consisting of four rhombuses with angles of 72° and 108° and one regular five-pointed star with an angle of 36° and inscribed in a regular pentagon. Additionally, parquet variants are presented, in which the gaps between regular pentagons, formed after the first iteration, are filled with the corners of regular five-pointed stars with an angle of 36° at the top. We assume that our further research will be directed to the investigation of a mosaic that has neither translation nor rotation symmetry and at the same time maintain a law-governed nature in the arrangement of tiles.

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ПАРКЕТИ НА ОСНОВІ ФРАКТАЛЬНОГО РОЗШИРЕННЯ ПРАВИЛЬНОГО П’ЯТИКУТНИКА

Розглянемо розгортку додекаедра - правильного багатогранника, поверхня якого складається з дванадцяти правильних п’ятикутників. Уявімо розгортку додекаедра у вигляді двох груп багатокутників, що складаються з 6 правильних п'ятикутників, один з яких розташовується в центрі групи, а інші примикають до його сторін. Однак найзначніше є те, що кожна група з 6 правильних п’ятикутників утворює фігуру, що вписується в правильний п'ятикутник. Звідси випливає, що кожну групу з 6 правильних п’ятикутників можна розглядати як результат фрактального розширення правильного п’ятикутника, що є у її центрі. Більше того, якщо до кожної сторони фігури, в яку вписується група з 6 правильних п’ятикутників, приставити таку ж фігуру, то отримаємо ще одну *фігуру, подібну до вихідного правильного п’ятикутника. Повторимо попередні дії нескінченне число разів і отримаємо фігуру, подібну до правильного п’ятикутника.*

Вперше застосовано фрактальне розширення правильного п’ятикутника до замощення площини, пропуски якої усуваються фігурами, що є комбінаціями ромба з кутами 36° і 144°, ромба з кутами 72° і 108° та правильної п’ятикутної зірки. Показано, що різноманітність багатокутників, що застосовуються для усунення пропусків, забезпечує паркетам у вигляді фрактального розширення правильного п’ятикутника більш високу художню цінність порівняно з паркетами, складеними за допомогою фрактального розширення квадрата. Представлено варіант паркету, у якого пропуски між правильними п’ятикутниками, що утворюються після виконання першої ітерації, заповнюються ромбами з кутами 36° і 144°. Показано подібність фігур, які заповнюють пропуски, що з’являються після виконання кожної ітерації, ромбу з кутами 36° і 144°, і подібність паркету, отриманого після четвертої ітерації та усіченого правильним п’ятикутником, фігурі, що складається з чотирьох ромбів з кутами 72° і 108° та однієї правильної п’ятикутної зірки з кутом 36° і вписується в правильний п’ятикутник. Крім того, представлено варіанти паркету, у якого пропуски між правильними п’ятикутниками, що утворюються після виконання першої ітерації, заповнюються кутами правильних п’ятикутних зірок з кутом 36° при вершині.

Ключові слова: мозаїки, паркети з ромбів та правильних п’ятикутних зірок, фрактальне розширення правильного п’ятикутника.